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Abstract

This paper addresses the toll pricing framework for the first-best pricing with logit-based stochastic user equilibrium (SUE) constraints. The first-best pricing is usually known as marginal-cost toll, which can be obtained by solving a traffic assignment problem based on the marginal cost functions. The marginal-cost toll, however, has rarely been implemented in practice, because it requires every specific link on the network to be charged. Thus, it is necessary to search for a substitute of the marginal cost pricing scheme, which can reduce the toll locations but still minimize the total travel time. The toll pricing framework is the set of all the substitute toll patterns of the marginal cost pricing. Assuming the users' route choice behavior following the logit-based SUE principle, this paper has first derived a mathematical expression for the toll pricing framework. Then, by proposing an origin-based variational inequality model for the logit-based SUE problem, another toll pricing framework is built, which avoids path enumeration/storage. Finally, the numerical test shows that many alternative pricing patterns can inherently reduce the charging locations and total toll collected, while achieving the same equilibrium link flow pattern.

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TOLL PRICING FRAMEWORK UNDER LOGIT-BASED STOCHASTIC USER EQUILIBRIUM CONSTRAINTS

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ABSTRACT

This paper addresses the toll pricing framework for the first-best pricing with logit-based stochastic user equilibrium (SUE) constraints. The first-best pricing is usually known as marginal-cost toll, which can be obtained by solving a traffic assignment problem based on the marginal-cost functions. The marginal-cost toll, however, has rarely been implemented in practice, since it requires every specific link on the network to be charged. Thus, it is necessary to search for a substitute of the marginal-cost pricing scheme, which can reduce the toll locations but still minimize the total travel time. The toll pricing framework is the set of all the substitute toll patterns of the marginal-cost pricing. Assuming the users' route choice behavior following the logit-based SUE principle, this paper has first derived a mathematical expression for the toll pricing framework. Then, by proposing an origin-based variational inequality model for the logit-based SUE problem, another toll pricing framework is built, which avoids path enumeration/storage. Finally, the numerical test shows that many alternative pricing patterns can inherently reduce the charging locations and total toll collected, while achieving the same equilibrium link flow pattern.

Keywords: First-Best Pricing; Toll Pricing Framework; Stochastic User Equilibrium; Variational Inequality; Linear Duality Theory

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1. INTRODUCTION

Congestion pricing is taken as one of the most effective instruments by the network authority to mitigate traffic congestion in urban area (see Yang and Huang, 2005; Lawphongpanich et al., 2006; Small and Verhoef, 2007; Zhang et al., 2011; Li, et al., 2012; Zhong et al., 2012; to name a few). Among various congestion pricing regimes, the first-best pricing scheme has attracted much attention from researchers. Ever since the seminal work by Pigou in 1920, studies on first-best pricing are prosperous yet still prevailing, see a recent review by de Palma and Lindsey (2011).

The first-best pricing scheme aims to optimize a systematic index (e.g., minimize total travel time or maximize the total social benefit) without limiting the charging locations, and it is usually known as the marginal cost pricing. The first-best pricing solution can be obtained by solving a traffic assignment problem with the marginal cost functions (Yang and Huang, 1998; Yang, 1999). Despite its sound properties for theoretical studies, the practical implementation of the first-best pricing scheme has been largely restricted. This is mainly because (a) from the viewpoint of land transport authorities, the marginal cost pricing scheme defines toll charge on each particular link and thus requires unaffordable cost for implementation; and (b) the oppositions from all the network users are strong since introducing marginal cost pricing means “every road should pay” and inconvenience to their daily commuting. To cope with this issue, Bergendorff et al. (1997) have proposed the toll pricing framework for the marginal cost pricing. Suppose there exists such kind of alternative toll pattern(s) to replace the marginal cost pricing, yet still achieving the same network equilibrium with identical link flows. The toll pricing framework is the set of such toll patterns. Hence, these alternative toll patterns are still solutions to the first-best pricing problem. Based on the toll pricing framework, we can select the first-best toll patterns more suitable for practical implementations, for instance, the first-best toll with minimal toll booths/locations, the first-best toll with minimal total charges, etc.

Nevertheless, all the previous studies on toll pricing framework assume deterministic user equilibrium (DUE). DUE is impractical via assuming perfect information from the users on predicted travel times. Hence, to improve the realism of first-best pricing, the stochastic user equilibrium should be presumed. To the best of our knowledge, toll pricing framework under stochastic user equilibrium constraints is still an open question in the literature. This problem is hence addressed in this paper.

1.1 Literature Review

Dafermos and Sparrow (1971) first pointed out that the marginal cost pricing is not the only toll pattern for system optimum flows, and discussed about several goals to address the alternative tolls. Based on a variational inequality (VI) model for DUE traffic assignment problem, Bergendorff et al. (1997) showed that the DUE link flow is the optimum of a linear programming model. Then, the toll pricing framework was provided by virtue of the duality model for this linear programming model. Hearn and Ramana (1998) have further tested different pricing objectives based on the toll pricing framework, which gives various alternative first-best tolls. These seminal works on toll pricing framework for marginal cost pricing were later extended to the elastic demand case by Hearn and Yildirim (2002) as well as Yildirim and Hearn (2005). It should be pointed out that Larsson and Patriksson (1998) have discovered that in the case of elastic demand, the total toll revenue is constant/identical for all the alternative first-best tolls. Considering improving the practicality of first-best pricing, the first-best toll with minimal toll locations/booths is of primary significance. Bai and Rubin (2009) have proposed a more efficient solution algorithm via the Combinatorial Benders Cuts to solve the minimum toll booth problem.

In spite of its practical significance, the studies for toll pricing framework are insufficient. Moreover, all the previous studies assume DUE for the users' driving behavior. Thus, in this paper, a toll pricing framework in the logit-based SUE case is proposed, where the VI model for logit-based SUE flows is utilized.

The proposed logit-based toll pricing framework is defined on path flows requiring path enumeration/storage, which is highly tedious for practical large-scale networks. To further improve the practicality of the addressed problem, we then develop a link-based expression for the logit-based toll pricing framework via proposing an origin-based VI model for the logit-based SUE problem, which obviates path enumeration/storage.

The origin-based formulation and solution algorithms for traffic assignment were exploited by Bar-Gera (2002) in the context of DUE, which possesses a faster convergence speed and higher solution accuracy than the link-based DUE. In the case of logit-based SUE, a convex minimization model was proposed by Akamatsu (1997) in terms of origin-based link flows, which can be solved by the decent direction algorithms. Based on the same formulation, the work of Akamatsu (1997) was then extended by Lee et al. (2010) via providing two more efficient algorithms with better line search techniques. The minimization model proposed by Akamatsu (1997) for logit-based SUE is also used in this paper for building the origin-based VI model.

To sum up, the contributions of this paper are twofold: first, to the best of our knowledge, this paper takes an initial step to build a toll pricing framework with respect to logit-based SUE constraints. This SUE toll pricing framework also nests the DUE toll pricing framework as a special case. In addition, two modeling extensions are also briefly discussed: the cases of c-logit model as well as elastic demand (see Section 6 for details). Second, this study proposes a VI model for the origin-based logit-based SUE flows and accordingly developed a link-based expression for the toll pricing framework that avoids path enumeration/generation. As a consequence, the toll pricing framework can be described by a set of constraints, where the number of the constraints is bounded by a polynomial expression of the network size.

2. PROBLEM DESCRIPTION

2.1 Notation and Assumptions

Considering a strongly connected network, denoted by G , the attributes of this network are denoted by the following notation.

- N : Set of nodes
- A : Set of links, $|A|$ is the cardinality of set A
- W : Set of OD pairs, $|W|$ is the cardinality of W
- q_w : Travel demand between OD pair $w \in W$
- \mathbf{q} : Column vector of all the OD travel demands, $\mathbf{q} = (q_w, w \in W)^T$
- R_w : Set of all the paths between OD pair $w \in W$, and $|R_w|$ is the cardinality of R_w
- δ_{ar}^w : $\delta_{ar}^w = 1$ if path $r \in R_w$ between OD pair $w \in W$ traverses link $a \in A$, $\delta_{ar}^w = 0$ otherwise
- Δ_w : Link/path incidence matrix associated with OD pair $w \in W$, namely,

$$\Delta_w = (\delta_{ar}^w, a \in A, r \in R_w)$$
- Δ : Link/path incidence matrix for the entire network, $\Delta = (\Delta_w, w \in W)$
- f_{wr} : Traffic flow on path $r \in R_w$ between OD pair $w \in W$
- \mathbf{f}_w : Column vector of traffic flows on all the paths between OD pair $w \in W$, namely,

$$\mathbf{f}_w = (f_{wr}, r \in R_w)^T$$
- \mathbf{f} : Column vector of traffic flow on all the paths in the network, i.e., $\mathbf{f} = (\mathbf{f}_w, w \in W)^T$

Λ : OD pair/path incidence matrix, $\Lambda = (\delta_{wr}, w \in W, r \in R_w)$, where δ_{wr} equals 1 if path $r \in R_w$ and 0, otherwise

v_a : Traffic flow on link $a \in A$

\mathbf{v} : Column vector of all the link traffic flows, $\mathbf{v} = (v_a, a \in A)^T$

$t_a(v_a)$: Travel time on link $a \in A$, and it is a non-negative, monotonically increasing and continuously differentiable function.

$c_{wr}(\mathbf{v})$: Travel time on path $r \in R_w$, and $c_{wr}(\mathbf{v}) = \sum_{a \in A} t_a(v_a) \delta_{ar}^w$

$\mathbf{c}_w(\mathbf{v})$: Column vector of travel times on all the paths connecting OD pair $w \in W$

According to the flow conservation law, path flows, link flows and OD travel demands should fulfill the following equations:

$$\mathbf{v} = \Delta \mathbf{f} \quad (1)$$

$$\mathbf{q} = \Lambda \mathbf{f} \quad (2)$$

$$\mathbf{f} \geq 0 \quad (3)$$

Eqs. (1)-(3) define the feasible set for path flows and link flows, denoted by Ω_f and Ω_v , respectively.

2.2 Logit-based Stochastic User Equilibrium Flow

In the context of logit-based stochastic user equilibrium principle, it is assumed that the users' perceived travel time on each path $r \in R_w$ equals:

$$C_{wr}(\mathbf{v}) = c_{wr}(\mathbf{v}) + \zeta_{wr} \quad (4)$$

where ζ_{wr} denotes the users' perception error on the path travel time, which is a random term with zero mean and fixed variance (Sheffi, 1985). The perception error on all the paths between OD pair w follows the independent and identical Gumbel distribution. Then, the route choice problem becomes a decision making problem based on the options with random utilities. Such kind of decision making problem can be analyzed by the discrete choice model (Ben-Akiva and Lerman, 1985), thus the equilibrium flow on each path should be:

$$f_{wr} = q_w p_{wr}, r \in R_w, w \in W \quad (5)$$

where p_{wr} is the path choice probability:

$$p_{wr} = \Pr(C_{wr} \leq C_{wl}, \forall l \in R_w \text{ and } l \neq r | \mathbf{c}_w), r \in R_w, w \in W \quad (6)$$

p_{wr} implies the probability that path $r \in R_w$ is perceived as the shortest one between all the paths connecting OD pair $w \in W$. It is well recognized that for logit-based SUE model, p_{wr} can be obtained by the following explicit form:

$$p_{wr} = \frac{\exp(-\theta c_{wr})}{\sum_{r \in R_w} \exp(-\theta c_{wr})} \quad (7)$$

where θ is a constant parameter. A fixed point model was proposed by Daganzo (1983) as follows. Any path flow pattern $\mathbf{f} \in \Omega_f$ is regarded as the equilibrium flow, if the flow on each path $r \in R_w$ can satisfy the following fixed point model:

$$f_{wr} = q_w p_{wr}(\mathbf{c}_w(\mathbf{v})), r \in R_w, w \in W \quad (8)$$

Based on the continuously differentiable and monotonically increasing travel time function $t_a(v_a)$, the existence and uniqueness of solution to Eq. (8) can be assured (Daganzo, 1983). The logit-based SUE problem can be solved by the Method of Successive Average (MSA) incorporating any stochastic network loading algorithm (Sheffi, 1985). The MSA is extended by Liu et al. (2009) via using adaptive step sizes.

2.3 Toll Pricing Framework

Suppose there are link-based toll charge on all of or part of the links on the network, and \bar{A} denotes the set of toll links (locations of tollbooths), evidently $\bar{A} \subset A$. Let $\boldsymbol{\tau} = (\tau_a \geq 0, a \in A)^T$ denote the column vector of toll charges on the network, where $\tau_a \geq 0$ on the toll links $a \in \bar{A}$ and $\tau_a \equiv 0$ on the un-tolled links. The existence of toll has affected the route choice decisions of the users, thus route choice problem should be analyzed based on the generalized link travel cost function:

$$\bar{t}_a(v_a) = t_a(v_a) + \tau_a, a \in A \quad (9)$$

Let $\mathbf{v}(\boldsymbol{\tau}) = (v_a(\boldsymbol{\tau}), a \in A)^T$ denote the equilibrium link flow in terms of the toll pattern $\boldsymbol{\tau}$, which can be obtained by solving a DUE or SUE problem based on the generalized link travel cost functions. For the convenience of analysis, the users' value-of-time is taken as 1.0 here.

Let $\boldsymbol{\tau}^*$ be the vector for the *first-best toll* in the context of DUE, which can give rise to the minimal total travel time on the network, i.e., $\sum_{a \in A} v_a t_a(v_a)$. It is well recognized that, the first-

best toll can be obtained by solving a traffic assignment problem based on the marginal cost function, see, e.g., Yang and Huang (2005):

$$\tilde{t}_a(v_a) = t_a(v_a) + v_a \frac{\partial t_a(v_a)}{\partial v_a}, a \in A \quad (10)$$

Hence, the equilibrium link flow associated with τ^* , denoted by $\mathbf{v}(\tau^*) = (v_a(\tau^*), a \in A)^T$, is the minimum of $\sum_{a \in A} v_a t_a(v_a)$, which is known as the system optimum (SO) link flow. As mentioned in the introduction section, Bergendorff et al. (1997) has built a set of alternative tolls for the first-best toll τ^* . Denoted by $\tilde{\tau}^*$, any pattern of alternative tolls can give rise to the same equilibrium link flows, namely $\mathbf{v}(\tau^*) = \mathbf{v}(\tilde{\tau}^*)$. Let $\bar{A}_{\tilde{\tau}}$ denote the set of toll links with respect to toll pattern $\tilde{\tau}$. The number of tollbooths for the alternative toll may be different from that of the first-best toll, i.e., $|\bar{A}_{\tilde{\tau}^*}| \neq |\bar{A}_{\tau^*}|$, where $|\bar{A}_{\tilde{\tau}^*}|$ denotes the cardinality of $\bar{A}_{\tilde{\tau}^*}$.

Let $\Phi(\tau^*)$ be the set of alternative tolls for the first-best toll τ^* , and $\Phi(\tau^*)$ is also termed as the Toll Pricing Framework (Yildirim and Hearn, 2005). Bergendorff et al. (1997) and Hearn and Ramana (1998) have shown that $\Phi(\tau^*)$ is a polyhedron, built based on the variational inequality model of DUE problem and the duality theory of linear programming. Thereby, solving a traffic assignment problem based on any vector $\tilde{\tau}^* \in \Phi(\tau^*)$ in the polyhedron can give rise to the SO link flow. Then, those alternative tolls with better practicality can be addressed from $\Phi(\tau^*)$, and implemented in practice. The following four objectives are commonly discussed in the literature (Hearn and Ramana, 1998):

MINTB: search for the alternative toll with minimal toll booths/locations.

$$\min_{\tau \in \Phi(\tau^*)} \sum_{a \in A} y_a \quad (11)$$

subject to:

$$\begin{aligned} \tau_a &\leq M y_a, \forall a \in A \\ y_a &\in \{0, 1\} \end{aligned} \quad (12)$$

where M is a positive large value.

MINSYS: minimize the total tolls collected

$$\min_{\tau \in \Phi(\tau^*)} \sum_{a \in A} \tau_a v_a(\tau^*) \quad (13)$$

MINMAX: minimize the maximal value of toll charges on the network

$$\min_{\tau \in \Phi(\tau^*)} z \quad (14)$$

subject to

$$\tau_a \leq z, \forall a \in A \quad (15)$$

MINDIFF: minimize the difference of each two tolls

$$\min_{\tau \in \Phi(\tau^*)} \max \{ \tau_a - \tau_b, a \in A, b \in A \} \quad (16)$$

The objective of this paper is to build the toll pricing framework $\Phi(\tau^*)$ for the first-best toll under the logit-based SUE constraints, and then get the alternative tolls corresponding to the objectives listed above.

3. LOGIT-BASED TOLL PRICING FRAMEWORK

3.1 First-best Toll under Logit-based SUE Constraints

Playing the same role as system optimum for DUE traffic assignment, a Stochastic System Optimum (SSO) was proposed by Maher et al. (2005) for the SUE traffic assignment. Let $S_w(\mathbf{v}) = E \left[\min_{r \in R_w} \{ C_{wr}(\mathbf{v}) \} \right]$, $w \in W$ denote the satisfaction function (Sheffi, 1985) for the logit-based SUE, then the SSO aims to minimize the total expected perceived cost on the network, which can be obtained by solving the following model:

$$\min_{\mathbf{f} \in \Omega_f} z = \sum_{w \in W} q_w S_w(\bar{\mathbf{c}}_w(\mathbf{v}) + \bar{\mathbf{d}}_w(\mathbf{f}_w)) - \sum_{w \in W} \sum_{r \in R_w} \bar{d}_{wr}(\mathbf{f}_w) f_{wr} \quad (17)$$

where $\bar{\mathbf{d}}_w(\mathbf{f}_w)$ is a dummy function, which is built to make any feasible flow pattern $\mathbf{f} = (f_{wr}, w \in W, r \in R_w)$ fulfill the SUE condition (5)(5), see Maher et al. (2005) for detailed descriptions of this dummy function.

It has been proven by Maher et al. (2005) that the marginal cost function can still give rise to the SSO, in terms of SUE; namely, solving an SUE problem based on the marginal cost function (10)(40) gives the SSO flows. Let $\tilde{\mathbf{v}}^* = (\tilde{v}_a^*, a \in A)^T$ denote the SSO link flows, the first-best toll then equals

$$\tau_a^* = \tilde{v}_a^* \frac{\partial t_a(\tilde{v}_a^*)}{\partial \tilde{v}_a^*}, a \in A \quad (18)$$

For the ease of presentation, the first-best toll $\boldsymbol{\tau}^* = (\tau_a^*, a \in A)^T$ under logit-based SUE constraints are termed as *logit-based first-best toll*. The objective of this paper is therefore, to build the toll pricing framework $\Phi(\boldsymbol{\tau}^*)$ for the logit-based first-best toll, which is elaborated in the following sub-section. It should be noted that the equivalence between first-best toll and marginal-cost toll, for logit-based SUE, was also derived by Yang (1999) based on two entropy-type minimization models.

3.2 Feasible Set of Alternative Tolls

Given any link-based toll charge pattern $\boldsymbol{\tau} = (\tau_a \geq 0, a \in \bar{A}; \tau_a \equiv 0, a \in A \setminus \bar{A})^T$, this sub-section can build the logit-based Toll Pricing Framework for $\boldsymbol{\tau}$. Following the methodology used in Bergendorff et al. (1997), to build the toll pricing framework relies on the variational inequality (VI) model. Therefore, the following VI model for the logit-based SUE traffic assignment problem is first adopted:

$$\sum_{w \in W} \sum_{r \in R_w} \left(\bar{c}_{wr}(\mathbf{f}^*) + \frac{1}{\theta} \ln f_{wr}^* \right) (f_{wr} - f_{wr}^*) \geq 0, \forall \mathbf{f} \in \Omega_f \quad (19)$$

where $\bar{c}_{wr}(\mathbf{f}^*)$ is the generalized travel cost on path $r \in R_w$, i.e.,

$$\bar{c}_{wr}(\mathbf{f}) = \sum_{a \in A} \bar{t}_a(v_a) \delta_{ar}^w \quad (20)$$

The generalized link travel cost function was given by Eq. (9).

Eq. (19) can be rewritten as follows, in terms of link travel costs:

$$\sum_{a \in A} \left(t_a(\mathbf{v}^*) + \tau_a + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} (\ln f_{wr}^*) \delta_{ar}^w \right) (v_a - v_a^*) \geq 0 \quad (21)$$

Define a vector function:

$$\mathbf{T}(\mathbf{v}, \boldsymbol{\tau}) = \left(t_a(\mathbf{v}) + \tau_a + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} (\ln f_{wr}) \delta_{ar}^w, a \in A \right)^T \quad (22)$$

Hence, the VI function in left-hand-side of Eq. (21) is in fact $\mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})$. For

$\forall \mathbf{v} = (v_a, a \in A)^T \in \Omega_v$, Eq. (21) is equivalent to

$$\mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \mathbf{v} \geq \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \mathbf{v}^* \quad (23)$$

It implies that $\mathbf{v}^* = (v_a^*, a \in A)^T$ is a solution of the following minimization problem:

$$\min Z_1 = \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \mathbf{v} = \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \Delta \mathbf{f} \quad (24)$$

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subject to

$$\Lambda \mathbf{f} = \mathbf{q} \quad (25)$$

$$\mathbf{f} \geq 0 \quad (26)$$

The dual problem for model ~~(24)(24)~~ ~~(26)(26)~~ can be provided as:

$$\max \bar{Z}_1 = \mathbf{q}^T \mathbf{y} \quad (27)$$

subject to

$$\Lambda^T \mathbf{y} \leq \Delta^T \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau}) \quad (28)$$

Since the primal minimization problem has an optimal solution $\mathbf{v}^* = (v_a^*, a \in A)^T$, then based on the Strong Duality Theory (Bertsimas and Tsitsiklis, 1997), its dual problem also has an optimal solution \mathbf{y}^* , such that:

$$\mathbf{q}^T \mathbf{y}^* = \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \mathbf{v}^* \quad (29)$$

Meanwhile, the optimal solution \mathbf{y}^* must be feasible, i.e.,

$$\Lambda^T \mathbf{y}^* \leq \Delta^T \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau}) \quad (30)$$

According to Bergendorff et al. (1997) and Hearn and Ramana (1998), Eqs. ~~(29)(29)~~ and ~~(30)(30)~~ define the polyhedron for alternative tolls. Namely, any vector $(\boldsymbol{\tau}, \mathbf{y})^T$ that can satisfy Eqs. ~~(29)(29)~~ and ~~(30)(30)~~ will give rise to the link flows \mathbf{v}^* .

However, the above model is defined on path flows, thus its solution method requires path enumeration/storage. In order to avoid path enumeration and take use of the more efficient link-based methods, another logit-based toll pricing framework is proposed in the following section, based on the origin-based logit model.

4. ORIGIN-BASED VI MODEL

For the ease of presentation, we use $(k, j) \in A$ to denote one link whose tail and head nodes are k and j , respectively. Use o and d to denote the origin and destination node of OD pair $w \in W$, thus the travel demand is rewritten as q_{od} . Let O denote the set of origins on the network. For each origin $o \in O$, it may be used by different OD pairs, and let $D(o)$ denote the set of destinations associated with the origin $o \in O$. Then, we can obtain the total demand produced from $o \in O$ by summing up all the demand of OD pairs originating from o . Denoted by q^o , the total demand originating from $o \in O$ equals:

$$q^o = \sum_{d \in D(o)} q_{od} \quad (31)$$

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The term origin-based link flow is then introduced, denoted by v_{kj}^o , which reflects the portion of demand q^o using link $(k, j) \in A$.

The following origin-based convex minimization model was proposed by Akamatsu (1997) for the logit-based SUE problem

$$\min Z_2 = \frac{1}{\theta} \sum_{o \in O} \left[\sum_{(k,j) \in A} v_{kj}^o \ln v_{kj}^o - \sum_{j \in N} \left(\left(\sum_{k \in N_h(j)} v_{kj}^o \right) \ln \left(\sum_{k \in N_h(j)} v_{kj}^o \right) \right) \right] + \sum_{(k,j) \in A} \int_0^{v_{kj}} t_{kj}(x) dx \quad (32)$$

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subject to

$$\sum_{k \in N_i(n)} v_{kn}^o - \sum_{j \in N_h(n)} v_{nj}^o = q_{od} \delta_{dn} - \sum_{d \in D(o)} q_{od} \delta_{on}, \forall o \in O, n \in N \quad (33)$$

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$$v_{kj} = \sum_{o \in O} v_{kj}^o, \forall (k, j) \in A \quad (34)$$

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$$v_{kj}^o \geq 0, \forall o \in O, \forall (k, j) \in A \quad (35)$$

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where $N_i(n)$ is the set of tails of those links heading to n , and $N_h(n)$ denotes the set of heads of those links emanating from node n . δ_{dn} (δ_{on}) reflects the topological incidence relationships between node n and the destination (origin), i.e. $\delta_{dn} = 1$ ($\delta_{on} = 1$) if n is the destination (origin).

When considering the toll charges on each link, the objective function (32) should be rewritten as

$$\min Z_3 = \frac{1}{\theta} \sum_{o \in O} \left[\sum_{(k,j) \in A} v_{kj}^o \ln v_{kj}^o - \sum_{j \in N} \left(\left(\sum_{k \in N_i(j)} v_{kj}^o \right) \ln \left(\sum_{k \in N_i(j)} v_{kj}^o \right) \right) \right] + \sum_{(k,j) \in A} \int_0^{v_{kj}} (t_{kj}(x) + \tau_{kj}) dx \quad (36)$$

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Since the objective function Z_3 is continuously differentiable and the feasible set defined by Eqs. (33) and (35) is closed and convex, this minimization model for logit-based SUE problem is equivalent to:

$$\nabla Z_3(\mathbf{v}^{o*})^T (\mathbf{v}^o - \mathbf{v}^{o*}) \geq 0 \quad (37)$$

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where $\mathbf{v}^o = (v_{kj}^o, o \in O, (k, j) \in A)^T$ denotes the vector for origin-based link flows (Nagurney, 1993). Eq. (37) can be rewritten as

$$\sum_{o \in O} \sum_{(k,j) \in A} \left(t_{kj}(\mathbf{v}^*) + \tau_{kj} + \frac{1}{\theta} \left(\ln v_{kj}^{o*} - \ln \sum_{l \in N_i(j)} v_{lj}^{o*} \right) \right) (v_{kj}^o - v_{kj}^{o*}) \geq 0 \quad (38)$$

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Then, similar to Eq. ~~(22)~~(22), a vector function is defined for the origin-based model as follows:

$$\hat{\mathbf{T}}(\mathbf{v}^o, \boldsymbol{\tau}) = \left(t_{kj}(\mathbf{v}^o) + \tau_{kj} + \frac{1}{\theta} \left(\ln v_{kj}^o - \ln \sum_{l \in N_r(j)} v_{lj}^o \right), o \in O, (k, j) \in A \right)^T \quad (39)$$

A vector expression of Eq. (38) then would be

$$\hat{\mathbf{T}}(\mathbf{v}^{o*}, \boldsymbol{\tau})^T (\mathbf{v}^o - \mathbf{v}^{o*}) \geq 0 \quad (40)$$

Therefore, $\mathbf{v}^{o*} = (v_{kj}^{o*}, o \in O, (k, j) \in A)^T$ is the unique solution of the following minimization problem:

$$\min \hat{\mathbf{T}}(\mathbf{v}^{o*}, \boldsymbol{\tau})^T \mathbf{v}^o \quad (41)$$

subject to

$$\Delta^o \mathbf{v}^o = \Lambda^o \mathbf{q} \quad (42)$$

$$\mathbf{v}^o \geq 0 \quad (43)$$

where Eq. (42) is a vector expression of constraints (33). $\Delta^o (\Lambda^o)$ is the matrix for the incidence relationships between the origin-based flow at each node and the origin-based link flow (OD demand).

Likewise to the derivations in Section 3.1, the polyhedron for alternative tolls $(\boldsymbol{\tau}, \mathbf{y})^T$ of the origin-based model can be obtained as:

$$\begin{aligned} (\Lambda^o \mathbf{q})^T \mathbf{y} &= \hat{\mathbf{T}}(\mathbf{v}^{o*}, \boldsymbol{\tau})^T \mathbf{v}^{o*} \\ \Delta^{oT} \mathbf{y} &\leq \hat{\mathbf{T}}(\mathbf{v}^{o*}, \boldsymbol{\tau}) \end{aligned} \quad (44)$$

Evidently, to solve any minimization problem over the set defined by Eq. (44) does not require path generation/storage. For the readers' interests, the cardinality of various terms in Eq. (44) is provided as follows:

$$\mathbf{y}: |O||N| \times 1$$

$$\mathbf{q}: |W| \times 1$$

$$\mathbf{v}^o: |O||A| \times 1$$

$$\hat{\mathbf{T}}(\mathbf{v}^{o*}, \boldsymbol{\tau}): |O||A| \times 1$$

$$\Delta^o: |O||N| \times |O||A|$$

$$\Lambda^o: |O||N| \times |W|$$

Hence, the number of constraints in the logit-based toll pricing framework, Eq. (44), is 1 (equation) plus $|O||A|$ (inequality), and the number of variables is $|O||N|$ plus $|A|$.

The above two sections provide two logit-based toll pricing frameworks for any toll pattern $\tau = (\tau_a \geq 0, a \in \bar{A}; \tau_a \equiv 0, a \in A \setminus \bar{A})^T$, which are also effective for the marginal-cost toll $\tau^* = (v_a^* t_a(v_a^*), a \in A)^T$, where $v_a^*, a \in A$ denote the equilibrium link flows.

5. NUMERICAL EXAMPLE

To further test the logit-based toll pricing framework, a numerical example is adopted in this section. As shown in Figure 1, this example has 7 nodes, 11 links and 4 OD pairs.

(Figure 1 should be inserted around here)

The travel demands are tabulated in Table 1 below.

(Table 1 should be inserted around here)

The travel time function on each link follows the Bureau of Public Roads (BPR) type function:

$$t_a(v_a) = t_a^0 \left(1.0 + 0.15 \left(\frac{v_a}{C_a} \right)^4 \right), a \in A \quad (45)$$

where t_a^0 denotes the free flow travel time and C_a is the capacity of each link. The specific value of t_a^0 and C_a on each link are provided in Table 2.

(Table 2 should be inserted around here)

The marginal-cost toll can be obtained by solving a standard logit-based SUE problem based on the following marginal cost function:

$$\tilde{t}_a(v_a) = t_a(v_a) + v_a \frac{\partial t(v_a)}{\partial v_a} = t_a^0 \left(1.0 + 0.15 \left(\frac{v_a}{C_a} \right)^4 \right) + v_a t_a^0 \left(\frac{0.6}{C_a} \times \left(\frac{v_a}{C_a} \right)^3 \right) \quad (46)$$

The value of dispersion parameter θ in Eq. (7) is taken as 0.01 for this example, and the value-of-time in this paper is taken as 1.0 cent/second. For the sake of origin-based toll

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pricing framework, the origin-based approach is adopted to solve the logit-based SUE problem. Based on the convex minimization model in Akamatsu (1997), the two efficient partial linearization methods proposed by Lee et al. (2010) can be adopted to solve the origin-based logit-based SUE flows $\mathbf{v}^{o*} = (v_{kj}^{o*}, o \in O, (k, j) \in A)^T$. Then, summing up the optimal origin-based link flows gives the SUE link flows, namely, $v_a^* = \sum_{o \in O} v_a^{o*}, a \in A$. The optimal SUE link flows and marginal-cost tolls are tabulated in Table 3.

(Table 3 should be inserted around here)

Table 3 clearly shows that the marginal-cost toll on each link is larger than zero, thus the establishment of toll booths on each link is necessitated, which requires huge construction and operation costs and also large human resources, causing a great burden on the local government and operating companies. Moreover, the large value of toll charges would induce strong oppositions from the network users. Hence, based on the optimal marginal-cost toll, the origin-based toll pricing framework is produced in light of Eqs. (44). The cardinality of origins and links, $|O|$ and $|A|$, are 4 and 11 respectively for this example. Hence, the total number of constraints for the toll pricing framework is 45. Due to the space limit, the detailed expressions of these 45 constraints are omitted from here. Then, the four pricing objectives addressed from Eqs. (11) to (16) are sequentially tested, which gives four minimization models. These four models can be easily transferred to linear programming models, which are convenient to be solved by the commercial solvers, e.g., CPLEX.

(Table 4 should be inserted around here)

Table 4 gives the solutions of the four minimization models, which are four different patterns of the alternative tolls to the marginal-cost pricing. For the readers' interest, the values of objective function and total toll revenue for each scenario are also provided in Table 4. Clearly, among all the five toll patterns listed in Tables 3 and 4, the MINTB has the fewest toll booths/locations; MINSYS takes the smallest total toll revenue; MINMAX has the minimal maximal-toll, which is 30.25; and MINDIFF shows the most balanced toll pattern. An interesting phenomenon is that the MINSYS also has the minimal number of toll booths

that equals MINTB in this example, and this indicates that the optimal solution for MINTB may not be unique.

We can see from Table 4 that: first, for both MINTB and MINSYS, the number of charging locations ($|\bar{A}|=5$) is less than one half of the marginal-cost toll ($|\bar{A}|=11$), which has largely reduced the construction/operation cost of the toll booths; for instance, for the Electronic Road Pricing system in Singapore, the construction cost of one three-lane toll gantry is 1.5 million Singapore-Dollars in 2008. Second, the total toll revenue for marginal-cost toll is 1614961.74, which is three times higher than that of the MINSYS toll. Thus, the users' financial burden has been largely ameliorated, and it would ameliorate the public oppositions to congestion pricing in many cities globally. These phenomena imply that the toll pricing framework would be a great impetus for the practical implementations of first-best pricing.

(Table 5 should be inserted around here)

To further assess the robustness of the toll pricing framework, and also to validate the equivalence between marginal-cost tolls and the four scenarios of alternative tolls, a standard logit-based SUE traffic assignment based on the generalized travel time function (9) with respect to each toll pattern is conducted. The SUE link flows are provided in Table 5, which shows that the SUE link flows of these five scenarios are nearly identical. The data in Table 5 then numerically validates the robustness of the proposed toll pricing framework, and it also reveals that the logit-based first-best toll is not unique, which provides more freedom for the practical implementations.

6. TWO EXTENSIONS OF THE TOLL PRICING FRAMEWORK

Two important extensions are briefly covered in this section to the first-best toll pricing framework under logit-based SUE constraints, which are c-logit SUE and elastic demand. It can be seen that for both the path-based and origin-based toll pricing framework (in Sections 3 and 4, respectively), an important step for the modeling methodology is to build a suitable VI model. Thus, the VI models for the cases of c-logit SUE and SUE with elastic demand are of considerable importance for the following two extensions. It should be noted that for simplicity of analysis, only the path-based models are considered here.

6.1 Extension to the C-logit SUE Case

Despite possessing a concise close-form expression for the choice probability function, the logit-based SUE is well recognized to have an intrinsic drawback: due to its independent assumption on the perception error, it is not valid to analyze paths with overlaps. To overcome this drawback of logit-based SUE, a large many studies have been conducted in the literature, including nested-logit, paired-combinational logit, c-logit, among many others, see a recent review and study by Zhou et al. (2012) for more details. Here, we take c-logit model as an example for the extension of logit-based SUE to consider the path correlation/overlaps.

First, the following VI model is proposed for the c-logit model with tolls, according to the VI model in Zhou et al. (2012) for c-logit model:

$$\sum_{w \in W} \sum_{r \in R_w} \left(\bar{c}_{wr}(\mathbf{f}^*) + cf_{wr} + \frac{1}{\theta} \ln f_{wr}^* \right) (f_{wr} - f_{wr}^*) \geq 0, \forall \mathbf{f} \in \Omega_f \quad (47)$$

where cf_{wr} is a commonality factor of route $r \in R_w$, which is flow independent and proportional to the degree of similarities between route r and other paths in R_w . Then, a link-based expression can be obtained based on this VI model:

$$\sum_{a \in A} \left(t_a(\mathbf{v}^*) + \tau_a + \sum_{w \in W} \sum_{r \in R_w} cf_{wr} + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} (\ln f_{wr}^*) \delta_{ar}^w \right) (v_a - v_a^*) \geq 0 \quad (48)$$

Similar to Eq. ~~(22)~~(22), we define a vector function:

$$\bar{\mathbf{T}}(\mathbf{v}, \boldsymbol{\tau}) = \left(t_a(\mathbf{v}) + \tau_a + \sum_{w \in W} \sum_{r \in R_w} cf_{wr} + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} (\ln f_{wr}) \delta_{ar}^w, a \in A \right)^T \quad (49)$$

Accordingly, the same procedure in Section 3 from Eq. ~~(23)~~(23) to Eq. ~~(30)~~(30) can be followed to build the toll pricing framework for any toll pattern $\boldsymbol{\tau} = (\tau_a \geq 0, a \in \bar{A}; \tau_a \equiv 0, a \in A \setminus \bar{A})^T$ in the context of c-logit model.

6.2 Extension to the Elastic Demand Case

For traffic assignment problem with elastic/variable demand, the OD demand \mathbf{q} is taken as a decreasing function of the travel impedance. In the DUE case, the OD travel impedance is reflected by the travel cost on the paths with positive flows (Sheffi, 1985), while in SUE case it is the satisfaction $S_w(\mathbf{v}) = E \left[\min_{r \in R_w} \{C_{wr}(\mathbf{v})\} \right], w \in W$; see the studies by Cantarella (1997), Meng and Liu (2012b) for more discussions about SUE with elastic demand. Hence, the travel demand in the case of logit-based SUE with elastic demand is defined by:

$$q_w = D_w(S_w) \in [0, \bar{q}_w] \quad (50)$$

where $D_w(\cdot)$ is the decreasing demand function, and \bar{q}_w is the upper-bound of travel demand q_w , which is decided by the total population and car-ownership in its origin zone. Then, the following VI model is proposed for the logit-based SUE with elastic demand and tolls:

$$\sum_{w \in W} \sum_{k \in R_w} \left(\bar{c}_{wk}(\mathbf{f}^*) + \frac{1}{\theta} \ln f_{wk}^* \right) (f_{wk} - f_{wk}^*) - \sum_{w \in W} \left(D^{-1}(q_w^*) + \frac{1}{\theta} (\ln q_w^* + 1) \right) (q_w - q_w^*) \geq 0, \forall \mathbf{f}, \mathbf{q} \in \Omega_{fq} \quad (51)$$

where $\Omega_{fq} = \left\{ v_a = \sum_{w \in W} \sum_{k \in R_w} f_{wk}, a \in A, \sum_{k \in R_w} f_{wk} = q_w, q_w = D(S_w), q_w \in [0, \bar{q}_w], w \in W, f_{wk} \geq 0 \right\}$ is the feasible set for the path flow and demand vector \mathbf{f}, \mathbf{q} . Considering the strict monotonicity of link travel time functions as well as demand functions, the Stochastic System Optimum flow pattern $\mathbf{f}^*, \mathbf{q}^*$ is unique (Liu, 2011).

Define a vector function $\mathbf{H}(\mathbf{q}) = \left(D^{-1}(q_w) + \frac{1}{\theta} (\ln q_w + 1), w \in W \right)^T$. Then, the same methodology for the case of logit-based SUE with fixed demand in Section 3 is followed, namely, $\mathbf{f}^*, \mathbf{q}^*$ is the solution of the following minimization problem:

$$\min Z_4 = \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \mathbf{v} - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q} = \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \Delta \mathbf{f} - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q} \quad (52)$$

subject to:

$$\Lambda \mathbf{f} - \mathbf{q} = \mathbf{0} \quad (53)$$

$$\mathbf{f} \geq \mathbf{0} \quad (54)$$

$$-\mathbf{q} \geq -\bar{\mathbf{q}} \quad (55)$$

$$\mathbf{q} \geq \mathbf{0} \quad (56)$$

And its dual problem is:

$$\max \bar{Z}_4 = -\bar{\mathbf{q}}^T \boldsymbol{\lambda} \quad (57)$$

subject to:

$$\Lambda^T \mathbf{y} \leq \Delta^T \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau}) \quad (58)$$

$$-\mathbf{y} - \boldsymbol{\lambda} \leq -\mathbf{H}(\mathbf{q}^*) \quad (59)$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \quad (60)$$

Then, the following expression can be provided for the toll pricing framework, based on the strong duality theory and feasibility of the variables:

$$\mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \Delta \mathbf{f}^* - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q}^* = -\bar{\mathbf{q}}^{*T} \boldsymbol{\lambda}^* \quad (61)$$

$$\Lambda^T \mathbf{y}^* \leq \Delta^T \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau}) \quad (62)$$

$$-\mathbf{y}^* - \boldsymbol{\lambda}^* \leq -\mathbf{H}(\mathbf{q}^*) \quad (63)$$

$$\boldsymbol{\lambda}^* \geq 0 \quad (64)$$

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Proposition 1. In the elastic demand case, the total toll revenue for the first-best alternative tolls is constant, i.e., all the toll patterns in the toll pricing framework produce the same total toll revenue.

Proof to Proposition 1:

In fact, the constraints (55) are always not binding. This is because the demand function is decreasing, and the travel impedance between OD pair w at equilibrium is definitely larger than zero, so optimal q_w^* is less than \bar{q}_w . Therefore, the optimal dual variable $\boldsymbol{\lambda}^* \equiv 0$, and hence the objective function of the dual problem in Eq. (57) is always zero. As per Eq. (61), we know that

$$\mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau})^T \Delta \mathbf{f}^* - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q}^* = -\bar{\mathbf{q}}^{*T} \boldsymbol{\lambda}^* = 0 \quad (65)$$

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For any two toll patterns, $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$, in the toll pricing framework $\Phi(\boldsymbol{\tau})$, we know that

$$\mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau}^1)^T \Delta \mathbf{f}^* - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q}^* = 0 = \mathbf{T}(\mathbf{v}^*, \boldsymbol{\tau}^2)^T \Delta \mathbf{f}^* - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q}^* \quad (66)$$

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Thus,

$$\begin{aligned} & \sum_{a \in A} \tau_a^1 v_a^* + \left(t_a(\mathbf{v}^*) + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} (\ln f_{wr}^*) \delta_{ar}^w, a \in A \right)^T \Delta \mathbf{f}^* - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q}^* \\ &= \sum_{a \in A} \tau_a^2 v_a^* + \left(t_a(\mathbf{v}^*) + \frac{1}{\theta} \sum_{w \in W} \sum_{r \in R_w} (\ln f_{wr}^*) \delta_{ar}^w, a \in A \right)^T \Delta \mathbf{f}^* - \mathbf{H}(\mathbf{q}^*)^T \mathbf{q}^* \end{aligned} \quad (67)$$

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It implies that

$$\sum_{a \in A} \tau_a^1 v_a^* = \sum_{a \in A} \tau_a^2 v_a^* \quad (68)$$

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which proves that all the toll patterns in the toll pricing framework $\Phi(\boldsymbol{\tau})$ for a feasible toll pattern $\boldsymbol{\tau} = (\tau_a \geq 0, a \in \bar{A}; \tau_a \equiv 0, a \in A \setminus \bar{A})^T$ will produce the same total toll revenue.

It should be noted that this finding in Proposition 1 is consistent with the study of Larsson and Patriksson (1998) as well as Yildirim and Hearn (2005) for the toll pricing framework in the case of DUE with elastic demand, which could provide interesting implications to the policy makers in practice, indicating that toll revenue should not be taken as an objective when setting the alternative first-best tolls.

7. CONCLUSIONS

This paper has proposed two toll pricing frameworks for the first-best toll under logit-based SUE constraints. The second toll pricing framework was built via proposing a VI model for the origin-based SUE link flows, thus it can obviate path enumeration/storage, making the toll pricing framework more suitable for practical implementation. The numerical experiment showed that: by solving different objectives subject to the toll pricing framework, substitutes/alternatives for the marginal-cost toll can be obtained. These alternatives can obtain the same link flows with the case of marginal-cost toll, yet largely reduce the number of toll locations and the total toll charges, which would highly improve the practical implementation of first-best tolls.

It should be pointed out that although the first-best toll is taken as a target in this paper, the proposed two toll pricing framework is effective for any other congestion pricing scheme, including the second-best pricing, area-wide pricing, cordon-based pricing, multimodal road pricing, or nonlinear pricing (see, e.g., Yin and Lou, 2009; Lawphongpanich and Yin, 2012; Meng and Liu, 2012a; Meng et al., 2012; Liu et al., 2013). Future efforts are needed to numerically test the performance of the alternative tolls in such cases.

An intrinsic connection is well recognized between the congestion pricing problem and capacity/side constrained traffic assignment problem (Larsson and Patriksson, 1998). When setting a link-based toll equal to the optimal Lagrangian multiplier of the link capacity constraint, it converts the capacity/side constrained traffic assignment problem to a standard traffic assignment problem. The toll pricing framework proposed in this paper can also be used to improve the practicality of such kind of congestion pricing problems under logit-based SUE constraints.

This paper focuses on the modeling side of the toll pricing framework for the first-best tolls under logit-based stochastic user equilibrium constraints. With the concise link-based expression provided by Eq. (44), it is straightforward to test the theoretical achievements of this paper on large-scale real transport networks. Future works are necessary to extend the numerical experiments on various large-scale networks as well as randomly generated networks.

This paper is an initial step for the study of toll pricing framework under SUE constraints. It is worthwhile to extend the findings in this paper along some more general directions, by considering the general SUE constraints, dynamic traffic networks, multi-user classes, multi-vehicle types as well as asymmetric link travel time functions, etc.

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Figure 1. Network structure for the numerical experiment

Table 1. OD demand

OD Pair	Travel Demand (vehicle/hour)
$1 \rightarrow 7$	6000
$2 \rightarrow 7$	5000
$3 \rightarrow 7$	5000
$6 \rightarrow 7$	4000

Table 2. Parameters in link travel time functions

Link No. a	Free-flow travel time (seconds) t_a^0	Capacity (vehicles/hour) C_a
1	60	4000
2	50	4000
3	60	4000
4	70	4000
5	60	2000
6	10	2000
7	50	3000
8	100	3000
9	110	4000
10	110	4000
11	150	4000

Table 3. SUE link flows and marginal-cost tolls

Link No. a	SUE link flows (vehicles/hour) v_a^*	Marginal-cost tolls (cents) τ_a^*
1	2887.57	9.78
2	2252.13	3.01
3	4694.19	68.28
4	4904.02	94.89
5	2461	82.53
6	2651.89	18.55
7	1806.62	3.95
8	2539	30.78
9	3887.12	58.86
10	3193.38	26.81
11	3321.29	42.78

Table 4. Alternative toll patterns

Link No. <i>a</i>	MINTB	MINSYS	MINMAX	MINDIFF
1	0.00	35.29	10.41	53.37
2	0.56	55.13	30.25	71.46
3	35.28	0.00	30.25	53.37
4	54.57	0.00	30.25	55.13
5	35.65	35.62	30.25	71.46
6	0.00	54.54	24.32	53.37
7	10.12	45.42	15.17	63.50
8	0.00	0.00	0.00	53.37
9	0.00	0.00	0.00	53.93
10	0.00	0.00	0.00	71.46
11	0.00	0.00	5.36	71.46
Objective value	5.00	540433.97	30.25	18.08
Total toll revenue	540498.65	540433.97	572649.21	2078814.23

Table 5. SUE link flows in each scenario

Link No. <i>a</i>	MINTB	MINSYS	MINMAX	MINDIFF	Marginal Cost Case
1	2886.16	2886.12	2886.15	2886.15	2887.57
2	2251.79	2251.80	2251.93	2251.93	2252.13
3	4694.63	4694.40	4694.43	4694.43	4694.19
4	4904.07	4904.30	4904.08	4904.08	4904.02
5	2460.80	2460.88	2461.07	2461.08	2461.00
6	2652.29	2652.50	2652.15	2652.15	2651.89
7	1808.47	1808.28	1808.28	1808.28	1806.62
8	2539.20	2539.12	2538.93	2538.92	2539.00
9	3886.92	3886.62	3886.77	3886.77	3887.12
10	3191.53	3191.72	3191.72	3191.72	3193.38
11	3322.85	3322.97	3323.00	3323.00	3321.29

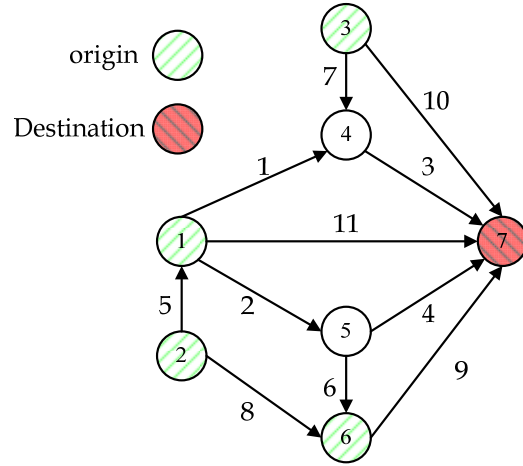


Figure 1. Network structure for the numerical experiment